IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 05, Issue 01 (January. 2015), ||V1|| PP 24-33

Bulk Demand (S, s) Inventory System with PH Distributions

Sundar. V^1, Rama. G^2, Ramshankar. R^3, Sandhya. R[4], Ramanarayanan. R^5
[1]*Senior Testing Engineer, ANSYS Inc., 2600, ANSYS Drive, Canonsburg, PA 15317, USA*
[2]*Independent Researcher B. Tech, Vellore Institute of Technology, Vellore, India*
[3]*Independent Researcher MS9EC0, University of Massachusetts, Amherst, MA, USA*
[4]*Independent Researcher MSPM, School of Business, George Washington University, Washington, D.C, USA.*
[5]*Professor of Mathematics, (Retired), Veltech, Dr. RR & Dr.SR Technical University, Chennai, India.*

Abstract: This paper studies two stochastic bulk demand (S, s) inventory models A and B both with k_1 phases PH demand occurrence and k_2 phases PH lead time distributions respectively. In the models the maximum storing capacity of the inventory is S units and the order for filling up the inventory is placed when the inventory level falls to s or below. The demands occur when the absorption occurs in the demand arrival PH process. Its size is random and is bounded above with distribution function depending on the phase from which the absorption occurs. Lead time for an order realization has phase type distribution with the supply size ≥ S and is bounded above with bounds depending on the phase from which the absorption occurs in the PH lead time distribution. When the inventory has no stock, demands wait for supply forming a queue. Only after clearing the waiting demands, the inventory is filled up. When an order is placed only after its realization, the next order can be placed. When the inventory is filled up due to an order realization, units received in excess are returned. In model A, the maximum of the maximum demand sizes in the k_1 phases is greater than the maximum of the k_2 supply sizes. In model B the maximum of the maximum demand sizes in the k_1 phases is less than the maximum of the k_2 supply sizes. Matrix partitioning method is used to study the inventory systems. The stationary probabilities of the inventory stock size, the probability of the waiting demand length, its expected values, its variances and probabilities of empty levels are derived for the two models using the iterated rate matrix. Numerical examples are presented for illustration.

Keywords: *Block Partition, Bulk Demand, Matrix Geometric Method, Phase Type Distribution, Supply after Lead Time.*

I. INTRODUCTION

In this paper two bulk demand (S, s) inventory systems are studied with phase type (PH) distributions. Models with PH distributions are very useful and important since PH distribution includes Exponential, Hyper exponential, Erlang and Coxian distributions as special cases and serves as suitable approximations for arbitrary general distributions as noted in Salah, Rachid, Abdelhakim and Hamid [1]. Inventory and queue models have been analyzed by many researchers. Thangaraj and Ramanarayanan [2] have studied two ordering level and unit demand inventory systems using integral equations. Jacob and Ramanarayanan [3] have treated (S, s) inventory systems with server vacations. Bini, Latouche and Meini [4] have studied numerical methods for Markov chains. Chakravarthy and Neuts [5] have discussed in depth a multi-server waiting model. Gaver, Jacobs and Latouche [6] have treated birth and death models with random environment. Latouche and Ramaswami [7] have studied Matrix Analytic methods in stochastic modeling. For matrix geometric methods and models one may refer Neuts [8]. Rama Ganesan, Ramshankar and Ramanarayanan [9] have analyzed M/M/1 bulk arrivals and bulk service queues under varying environment. Fatigue failure models using Matrix geometric methods have been treated by Sundar Viswanathan [10]. The models considered in this paper are general compared to existing inventory models. Here at each demand epoch, random numbers of units are demanded and the maximum number of units demanded may be different in different phases. When there is no stock in the inventory, after the lead time, realized orders can clear various number of waiting demands. Usually bulk arrival models have partitions based on M/G/1 upper-Heisenberg block matrix structure with zeros below the first sub diagonal. The decomposition of a Toeplitz sub matrix of the infinitesimal generator is required to find the stationary probability vector. Matrix geometric structures have not been noted as mentioned by William J. Stewart [11] and even in such models the recurrence relation method to find the stationary probabilities is stopped at certain level in most general cases indicating the limitations of such approach. Rama Ganesan and Ramanarayanan [12] have presented a special case where a generating function has been noticed in such a situation. But in this paper the partitioning of the matrix with blocks of size, which is the maximum of the maximum number of demands in all phases and the maximum of the order supply sizes in all phases together with PH phases, exhibits the matrix geometric structure for the (S, s) bulk demand inventory system with PH distributions. The (S, s) inventory systems of M/M/1 types with bulk demands, bulk supply and random environment have been treated by Sundar Viswanathan, Rama Ganesan, Ramshankar and Ramanarayanan in [13]. In this paper two models (A) and (B) of bulk demand (S, s) inventory systems with PH demand, PH lead time and infinite storage spaces for demands are studied using the block partitioning method to obtain matrix geometric results. In the models considered here, the demand sizes are bounded discrete random variables with distinct distributions corresponding to the phase from which the PH arrival process moves to absorption state. The lead time distribution is of PH type and the order is realized at the epoch at which the absorbing state is reached. The size of the supply is finite depending on the last phase before the absorption. When the inventory becomes full, units realized in excess are returned immediately. Always waiting demands are given priority and to be cleared before providing stocks for the inventory. When the waiting queue of demands is longer than the supply size then the entire supply is utilized for reducing the queue length. Model (A) presents the case when M, the maximum of all the maximum demand sizes is bigger than N, the maximum of order supply sizes. In Model (B), its dual case, N is bigger than M, is treated. In general in waiting line models, the state space of the system has the first co-ordinate indicating the number of customers in the system but here the demands in the system are grouped and considered as members of M sized blocks of demands (when M >N) or N sized blocks of demands (when N > M) for finding the rate matrix. The matrices appearing as the basic system generators in these two models due to block partitions are seen as block circulants. The stationary probability of the number of demands waiting for service, the expectation, the variance and the probability of various levels of the inventory are derived for these models. Numerical cases are presented to illustrate their applications. The paper is organized in the following manner. In section II the (S, s) inventory system with bulk demand and order clearance after the lead time is studied with PH distributions in which maximum M is greater than maximum N. Various performance measures are obtained. Section III treats the situation in which the maximum M is less than the maximum N. In section IV numerical cases are treated.

II. MODEL (A) MAXIMUM DEMAND SIZE M > MAXIMUM SUPPLY SIZE N

2.1Assumptions

(i) The time between consecutive epochs of bulk arrivals of demands has phase type distribution ($\underline{\alpha}$, T) where T is a matrix of order k_1 with absorbing rate $T_0 = -Te$ to the absorbing state k_1+1. When the absorption occurs, the next bulk demand arrival time starts instantaneously from a starting state as per the starting vector $\underline{\alpha}$ = $(\alpha_1, \alpha_2., ..., \ \alpha_{k_1})$ and $\sum_{i=1}^{k_1} \alpha_i$ = 1. Let φ be the invariant probability vector of the generator matrix $(T + T_0\underline{\alpha})$. When the absorption occurs in the PH arrival process due to transition from a state i to state k_1 +1, χ_i number of demands arrive with probabilities P (χ_i= j) = p_j^i for 1 ≤ j ≤ M_i and $\sum_{j=1}^{M_i} p_j^i$=1where M_i for1≤ i ≤$k_1$is the maximum size.

(ii)The maximum capacity of the inventory to store units is S. Whenever the inventory level falls to s or below, orders are placed for the supply of units for the inventory. Arriving demands are served till the inventory level falls to 0 after which the demands form a queue and wait for order realization.

(iii)The lead time distribution of an order has phase type distribution ($\underline{\beta}$, S) where S is a matrix of order k_2 with absorbing rate $S_0 = -Se$ to the absorbing state k_2+1 and the staring vector is $\underline{\beta}$ = $(\beta_1, \beta_2., ..., \ \beta_{k_2})$ where $\sum_{i=1}^{k_2} \beta_i$ = 1. Let ϕ be the invariant probability vector of the generator matrix $(S + S_0\underline{\beta})$. During the lead time of an order, another order cannot be placed. When absorption occurs due to a transition from state i to state k_2+1, an order is realized with the supply of constant $N_i \geq$ S units for 1≤ i ≤ k_2 and the waiting demands are served first and the balance if any becomes stock for the inventory. In case the inventory is filled up, the units which are in excess, if any are returned immediately. After the realization of an order if the inventory level becomes s or below s or the inventory is still empty with or without waiting demands, then the next order is immediately placed. When n demands are waiting for 0 ≤ n ≤ N_i-S at the order realization epoch, n waiting demands are cleared, the inventory is filled up and units in excess are immediately returned. When the waiting number of demands n at the order realization epoch is such that N_i-S < n < N_i all the n demands are cleared and N_i-n units become stocks for the inventory if N_i-n ≤ S and if N_i-n > S the inventory is filled up and the units in excess are returned. If n ≥ N_i demands are waiting when an order is realized, N_i demands are cleared reducing the waiting demand length to n-N_i.

(iv)The maximum of the maximum demand arrival size M=$\max_{1 \leq i \leq k_1} M_i$ is greater than the maximum of the order realization size N=$\max_{1 \leq j \leq k_2} N_j$.

2.3Analysis

The state of the system of the continuous time Markov chain X (t) under consideration is presented as follows. X(t)={(k, i) : for 0 ≤ k ≤ S-s-1 and 1 ≤ i ≤ k_1)}U{(0, k, i, j) ; for S-s ≤ k ≤ M-1; 1 ≤ i ≤ k_1; 1 ≤ j ≤ k_2} U{(n, k, i,j): for 0 ≤ k ≤ M-1; 1 ≤ i ≤ k_1; 1 ≤ j ≤ k_2 and n ≥ 1}. (1)

The chain is in the state (k, i) when the number of stocks in the inventory is S –k for 0 ≤ k ≤ S-s-1 and the arrival phase is i for 1 ≤ i ≤ k_1. The chain is in the state (0, k, i, j) when the number of stocks in the inventory is S-k for S-s ≤ k ≤ S without any waiting demand, the arrival phase is i for 1 ≤ i ≤ k_1 and the order-supply phase is j for 1 ≤ j ≤ k_2. The chain is in the state (0, k, i, j) when the number of waiting demands is k-S for S+1 ≤ k ≤ M-1, arrival phase is i for 1 ≤ i ≤ k_1 and the order-supply phase is j for 1 ≤ j ≤ k_2.The chain is in the state (n, k, i, j) when the number of demands in the queue is n M + k- S, for 0 ≤ k ≤ M-1 and 1 ≤ n < ∞, arrival phase is i for 1 ≤ i ≤ k_1 and the order-supply phase is j for 1 ≤ j ≤ k_2. When the number of demands waiting in the system is r ≥ 1, then r is identified with the first two co-ordinates (n, k) where n is the quotient and k is the remainder for the division of r + S by M; r = M n + k - S for r ≥ 1, 0 ≤ n < ∞ and 0 ≤ k ≤ M-1. Let the survivor probabilities of arrivals χ_i be P(χ_i > m) = P_m^i = 1 - $\sum_{n=1}^{m} p_n^i$, for 1 ≤ m ≤ M_i -1 with $P_0^i = 1$, for all i, 1 ≤ i ≤ k_1 (2) The chain X (t) describing model has the infinitesimal generator Q_A of infinite order which can be presented in block partitioned form given below.

$$Q_A=\begin{bmatrix} B_1 & B_0 & 0 & 0 & . & . & . & \cdots \\ B_2 & A_1 & A_0 & 0 & . & . & . & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & . & . & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{3}$$

In (3) the states of the matrix are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots$. For the partition purpose the states in the first two sets of (1) are combined. The vector $\underline{0}$ is of type 1 x [$k_1(S-s) + k_1k_2$(M-S+s)] and $\underline{n}$ is of type 1 x(k_1k_2M). They are $\underline{0}$ = ((0,1),(0,2)…(0,k_1)…(S-s-1,1)…(S-s-1, k_1),(0,S-s,1,1),(0,S-s,1,2)…(0,S-s, k_1, k_2) …(0,S,1,1)…(0,S, k_1, k_2),(0,S+1,1,1)…(0,S+1,k_1, k_2)…(0,M-1,1,1)…(0,M-1,$k_1 k_2$)). For n > 0 the vector is $\underline{n}$=((n,0,1,1),(n,0,1,2)…(n,0,1,k_2),(n,0,2,1),(n,0,2,2)…(n,0,2,k_2),(n,0,3,1)…(n,0,k_1, k_2),(n,1,1,1)...(n,1,k_1, k_2),(n ,2,1,1)……(n,2, k_1, k_2)….(n,M-1,1,1), (n,M-1,1,2) …(n,M-1,1,k_2),(n,M-1,2,1)……(n,M-1,k_1, k_2)). The matrices B_1 *and* A_1 have negative diagonal elements, they are of orders $k_1(S-s) + k_1k_2$(M-S+s) and k_1k_2M respectively and their off diagonal elements are non- negative. The matrices A_0 *and* A_2 have nonnegative elements and are of orders k_1k_2M. The matrices B_0 *and* B_2 have non-negative elements and are of types [$k_1(S-s) + k_1k_2$(M-S+s)] x (k_1k_2M) and (k_1k_2M) x [$k_1(S-s) + k_1k_2$(M-S+s)] respectively and they are all given below. Let ⊕ *and* ⊗ denote the Kronecker sum and Kronecker products respectively. Let Q_1'=T⊕S = (T⊗I_{k_2}) + (I_{k_1}⊗S) (4)

where I indicates the identity matrices of orders given in the suffixes and Q_1' is of order k_1k_2. Let T_0 = $(t_0^1, t_0^2, \ldots t_0^{k_1})'$ be the column vector of absorption rates in PH arrival process. Let S_0 = $(s_0^1, s_0^2, \ldots s_0^{k_2})'$ be the column vector of absorption rates concerning the PH lead time distribution. Let $T_{0j} = (t_0^1 p_j^1, t_0^2 p_j^2, \ldots, t_0^{k_1} p_j^{k_1})'$ for1 ≤ j ≤ M (5)

where $t_0^i p_j^i$ is the rate of absorption from state i to state k_1+1 when j demands occur for 1≤ i ≤ k_1 and for 1 ≤ j ≤ M. Let the matrix $\Lambda_j = [T_{0j}\,\underline{\alpha}\,] \otimes I_{k_2}$ *for* $1 \le j \le M$, (6)

$S_{0j} = \left(s_0^1 \delta_{j,N_1}, s_0^2 \delta_{j,N_2}, \ldots, s_0^{k2} \delta_{j,N_{k_2}}\right)'$ for1 ≤ j ≤ N ; S ≤ N_i ≤ N and 1 ≤ i ≤ k_2 (7)

where $\delta_{i,j}$ = 1 if i = j and $\delta_{i,j}$ = 0 if i ≠ j, 1 ≤ i,j ≤ N. S_{0j} is a 0 column vector for j ≠ N_i for 1 ≤ i ≤ k_2. S_{0N_i} is a column vector with only one non-zero element $(S_{0N_i})_i$=s_0^i for 1 ≤ i ≤ k_2 and $(S_{0N_i})_j$=0 if i ≠ j for 1 ≤ i, j ≤ k_2.

Let $U_j = I_{k_1} \otimes \left[S_{0j}\,\underline{\beta}\right]$ *for* $1 \le j \le N$ *with order* k_1k_2;

$U_j' = I_{k_1} \otimes [S_{0j}]$ *for* $1 \le j \le N$ *with type* $k_1k_2 x\, k_1$;

Λ_j' =$\left[T_{0j}\,\underline{\alpha} \otimes \underline{\beta}\right]$ *for* $1 \le j \le M$ with type k_1x (k_1k_2) ; Λ''_j =$[T_{0j}\,\underline{\alpha}\,]$ *for*$1 \le j \le S-s-1$ with order k_1;

$V_j = \sum_{i=j+1}^{N} U_j'$, for 1 ≤ j ≤ N-1. (8)

$$A_0=\begin{bmatrix} \Lambda_M & 0 & \cdots & 0 & 0 & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 \\ \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 & 0 \\ \Lambda_{M-3} & \Lambda_{M-2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M & 0 & 0 \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M \end{bmatrix} \tag{9}$$

$$A_2=\begin{bmatrix} 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_2 & U_1 \\ 0 & \cdots & 0 & 0 & U_N & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & U_N & U_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{10}$$

$$A_1 = \begin{bmatrix} Q_1' & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U_1 & Q_1' & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ U_2 & U_1 & Q_1' & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_N & U_{N-1} & U_{N-2} & \cdots & Q_1' & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U_N & U_{N-1} & \cdots & U_1 & Q_1' & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U_N & \cdots & U_2 & U_1 & Q_1' & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} & \cdots & Q_1' & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_1 & Q_1' \end{bmatrix} \quad (11)$$

$$B_1 = \begin{bmatrix} T & \Lambda''_1 & \Lambda''_2 & \cdots & \Lambda''_{S-s-1} & \Lambda'_{S-s} & \Lambda'_{S-s+1} & \cdots & \Lambda'_{M-N-2} & \Lambda'_{M-N-1} & \cdots & \Lambda'_{M-2} & \Lambda'_{M-1} \\ 0 & T & \Lambda''_1 & \cdots & \Lambda''_{S-s-2} & \Lambda'_{S-s-1} & \Lambda'_{S-s} & \cdots & \Lambda'_{M-N-3} & \Lambda'_{M-N-2} & \cdots & \Lambda'_{M-3} & \Lambda'_{M-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_{M-N-S+s-1} & \Lambda'_{M-N-S+s} & \cdots & \Lambda'_{M-(S-s+1)} & \Lambda'_{M-(S-s)} \\ V_{S-s-1} & U'_{S-s-1} & U'_{S-s-2} & \cdots & U'_1 & Q_1' & \Lambda_1 & \cdots & \Lambda_{M-N-S+s-2} & \Lambda_{M-N-S+s-1} & \cdots & \Lambda_{M-(S-s+2)} & \Lambda_{M-(S-s+1)} \\ V_{S-s} & U'_{S-s} & U'_{S-s-1} & \cdots & U'_2 & U_1 & Q_1' & \cdots & \Lambda_{M-N-S+s-3} & \Lambda_{M-N-S+s-2} & \cdots & \Lambda_{M-(S-s+3)} & \Lambda_{M-(S-s+2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \cdots & \vdots & \vdots \\ V_{N-1} & U'_{N-1} & U'_{N-2} & \cdots & U'_{N-S+s} & U_{N-S+s-1} & U_{N-S+s-2} & \cdots & U_1 & Q_1' & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U'_N & U'_{N-1} & \cdots & U'_{N-S+s+1} & U_{N-S+s} & U_{N-S+s-1} & \cdots & U_2 & U_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U'_N & \cdots & U'_{N-S+s+2} & U_{N-S+s+1} & U_{N-S+s} & \cdots & U_3 & U_2 & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & U_N & U_{N-1} & \cdots & Q_1' & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U_N & \cdots & U_1 & Q_1' \end{bmatrix} \quad (12)$$

$$B_0 = \begin{bmatrix} \Lambda'_M & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \Lambda'_{M-1} & \Lambda'_M & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda'_{M-(S-s)+1} & \Lambda'_{M-(S-s)+2} & \cdots & \Lambda'_M & 0 & \cdots & 0 \\ \Lambda_{M-(S-s)} & \Lambda_{M-(S-s)+1} & \ddots & \Lambda_{M-1} & \Lambda_M & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{S-s} & \Lambda_{S-s+1} & \cdots & \Lambda_M \end{bmatrix} \quad (13)$$

$$B_2 = \begin{bmatrix} 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_2 & U_1 \\ 0 & \cdots & 0 & 0 & U_N & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & U_N & U_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (14)$$

In B_2, the first $k_1(S-s) + k_1 k_2$(M-S+s-N) are 0 columns. In (14) the structure presents the case when M-N-S+s $\geq$ 0. When M-N < S-s, then in the column blocks from M-N to S-s $all\ U_j\ may\ be\ replaced\ by\ U'_j$. Similarly in the matrix B_1 the column blocks from 2 to S-s $all\ U_j\ may\ be\ replaced\ by\ U'_j$. The basic generator of the bulk queue which is concerned with only the demand and supply is a matrix of order $k_1 k_2$M given below in (17) where $Q''_A = A_0 + A_1 + A_2$ (15)

Its probability vector w gives, wQ''_A =0 and w. e = 1 (16)

It is well known that a square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a circulant matrix. It is very interesting to note that the matrix $Q''_A = A_0 + A_1 + A_2$ is a block circulant matrix where each block matrix is rotated one block to the right relative to the preceding.

$Q_A^{''} =$

$$\begin{bmatrix}
Q_1^{'}+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 \\
\Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3}+U_3 & \Lambda_{M-2}+U_2 \\
\Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4}+U_3 & \Lambda_{M-3}+U_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Lambda_{M-N+2}+U_{N-2} & . & \cdots & . & . & . & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} \\
\Lambda_{M-N+1}+U_{N-1} & . & \cdots & . & . & . & . & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N \\
\Lambda_{M-N}+U_N & . & \cdots & Q_1^{'}+\Lambda_M & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\
\Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\
\Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \Lambda_{M-N+2}+U_{N-2} & \cdots & Q_1^{'}+\Lambda_M & \Lambda_1 \\
\Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \cdots & \Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M
\end{bmatrix} \quad (17)$$

In (17), the first block-row of type k_1k_2x Mk_1k_2 is, $W = (Q_1^{'}+\Lambda_M, \Lambda_1, \Lambda_2, \ldots, \Lambda_{M-N-2}, \Lambda_{M-N-1}, \Lambda_{M-N}+U_N, \ldots, \Lambda_{M-2}+U_2, \Lambda_{M-1}+U_1)$ which gives as the sum of the blocks $(Q_1^{'}+\Lambda_M)+\Lambda_1+\Lambda_2+\ldots+\Lambda_{M-N-2}+\Lambda_{M-N-1}+\Lambda_{M-N}+U_N+\ldots+\Lambda_{M-2}+U_2+\Lambda_{M-1}+U_1=$ $(T+T_0\underline{\alpha}) \oplus (S+S_0\underline{\beta}) = (T+T_0\underline{\alpha})\otimes I_{k_2} + I_{k_1}\otimes(S+S_0\underline{\beta})$ whose stationary vector is $\varphi\otimes\phi$. This gives $\varphi\otimes\phi(Q_1^{'}+\Lambda_M)+\varphi\otimes\phi\sum_{i=1}^{M-N-1}\Lambda_i+\varphi\otimes\phi\sum_{i=1}^{N}(\Lambda_{M-i}+Ui)=0$. So $(\varphi\otimes\phi, \varphi\otimes\phi \ldots, \varphi\otimes\phi)$. W = 0 = $(\varphi\otimes\phi, \varphi\otimes\phi \ldots, \varphi\otimes\phi)$ W'. Since all blocks, in any block-row are seen somewhere in each and every column block structure (the matrix is block circulant), the above equation shows the left eigen vector of the matrix $Q_A^{''}$ is $(\varphi\otimes\phi, \varphi\otimes\phi, \ldots\ldots, \varphi\otimes\phi)$. Using (16), this gives probability vector $w=\left(\frac{\varphi\otimes\phi}{M}, \frac{\varphi\otimes\phi}{M}, \frac{\varphi\otimes\phi}{M}, \ldots\ldots, \frac{\varphi\otimes\phi}{M}\right)$ (18)

Neuts [8], gives the stability condition as, w $A_0\, e < w\, A_2\, e$ where w is given by (18). Taking the sum cross diagonally in the A_0 and A_2 matrices, it can be seen that the stability condition can be simplified as follows. w $A_0\, e=\frac{1}{M}\ \varphi\otimes\phi(\sum_{n=1}^{M} n\Lambda_n)e=\frac{1}{M}\ (\sum_{n=1}^{M} n(\varphi\otimes\phi\Lambda_n)e = \frac{1}{M}\ (\sum_{n=1}^{M} n(\varphi\otimes\phi)(T_{0n}\underline{\alpha} \otimes I_{k_2})e$ $=\frac{1}{M}\sum_{n=1}^{M} n\,\varphi\,[T_{0n}\underline{\alpha}\,]e\otimes\phi e = \frac{1}{M}\sum_{n=1}^{M} n\sum_{j=1}^{k_1}\varphi_j\, t_0^j p_n^i = \frac{1}{M}\sum_{j=1}^{k_1}\varphi_j\, t_0^j E(\chi_j) < w\, A_2\, e = \frac{1}{M}\ \varphi\otimes\phi(\sum_{n=1}^{N} nU_n)e$ = $\frac{1}{M}\ (\sum_{n=1}^{N} n(\varphi\otimes\phi U_n)e$ = $\frac{1}{M}\ (\sum_{n=1}^{N} n(\varphi\otimes\phi)(I_{k_1}\otimes S_{0n}\underline{\beta})e$ = $\frac{1}{M}\sum_{n=1}^{N} n\,\varphi\, e\otimes\phi S_{0n}\underline{\beta} e$ = $\frac{1}{M}\sum_{n=1}^{N} n\sum_{j=1}^{k_2}\phi_j\, s_0^j\,\delta_{n,N_j} = \frac{1}{M}\sum_{j=1}^{k_2}\phi_j\, s_0^j N_j$ where $\varphi_i$and ϕ_j are components of φ and ϕ respectively for $1 \le i \le k_1$ and $1 \le j \le k_2$. So the inequality for the steady state reduces to $\sum_{i=1}^{k_1}\varphi_i\ t_0^i E(\chi_i) < \sum_{i=1}^{k_2}\phi_i\ s_0^i N_i$ (19)

This is the stability condition for the bulk demand (S,s) inventory system with PH distributions when the maximum demand size in all arrival phases is greater than the maximum supply size in all supply phases. When (19) is satisfied, the stationary distribution of the queue length of waiting demands for units exists Neuts[8]

Let π (k, i) for $0 \le k \le$ S-s-1 and $1 \le i \le k_1$; π (0, k, i, j) for S- s $\le k \le$ M-1, $1 \le i \le k_1$and $1 \le j \le k_2$; π (n, k, i, j), for $0 \le k \le$ M-1, $1 \le i \le k_1$, $1 \le j \le k_2$ and n ≥ 1 be the stationary probability of the states in (1). Let π_0 = (π(0, 1)…π(0, k_1)…π(S-s-1, 1)…π(S-s-1, k_1) π(0, S-s, 1, 1), π(0, S-s, 1, 2)…π(0, M-1, k_1, k_2)) be of type 1 x $[k_1(S-s)+k_1k_2$(M-S+s)]. Let π_n= (π(n, 0, 1,1), π(n, 0,1, 2) … π(n, 0, 1,k_1), π(n, 0, 2, 1), π(n,0, 2, 2),…π(n,0,k_1, k_2), π(n,1,1,1),…….π(n, M-1, 1,1), π(n, M-1,1,2)…π(n, M-1,k_1k_2)) be of type 1 x k_1k_2M for n ≥ 1. The stationary probability vector $\pi = (\pi_0, \pi_1, \pi_3, \ldots)$ satisfies the equations πQ_A=0 and πe=1. (20)

From (20), it can be seen that $\pi_0 B_1 + \pi_1 B_2$=0. (21)

$\pi_0 B_0+\pi_1 A_1+\pi_2 A_2 = 0$ (22)

$\pi_{n-1}A_0+\pi_n A_1+\pi_{n+1}A_2 = 0$, for n ≥ 2. (23)

Introducing the rate matrix R as the minimal non-negative solution of the non-linear matrix equation A_0+RA_1+R^2A_2=0, (24)

it can be proved (Neuts [8]) that π_n satisfies the following. $\pi_n = \pi_1 R^{n-1}$ for n ≥ 2. (25)

Using (21), π_0 satisfies $\pi_0 = \pi_1 B_2(-B_1)^{-1}$ (26)

So using (22) , (26) and (25) the vector π_1 can be calculated up to multiplicative constant since π_1 satisfies the equation $\pi_1\,[B_2(-B_1)^{-1}B_0 + A_1 + RA_2]$ =0. (27)

Using (20), (25) and (26) it can be seen that $\pi_1[B_2(-B_1)^{-1}$e+(I-R)$^{-1}e]$ = 1. (28)

Replacing the first column of the matrix multiplier of π_1 in equation (27), by the column vector multiplier of π_1 in (28), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is π_1 and this gives along with (26) and (25) all the stationary probabilities of the system. The matrix R is iterated starting with$R(0) = 0$; and by finding $R(n+1)$=$-A_0A_1^{-1}$–$R^2(n)A_2A_1^{-1}$, n ≥ 0. (29)

The iteration may be terminated to get a solution of R at a norm level where $\|R(n+1)-R(n)\| <$ ε.

2.3. Performance Measures

(1) The probability of the demand length L = r > 0, P (L = r), can be seen as follows. Let n ≥ 0 and k for 0 ≤ k ≤ M-1 be non-negative integers such that r = n M + k - S. Then using (21) (22) and (23) it is noted that P (L=r) = $\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(n,k,i,j)$, where r = n M + k –S > 0.

(2) P (waiting demand length L = 0) = P (L = 0) = $\sum_{k=0}^{S-s-1}\sum_{i=1}^{k_1}\pi(\mathrm{k,i}) + \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{k=S-s}^{S}\pi(0,k,i,j)$ and for r > 0, P (Inventory level is r) = P (INV= r) = $\begin{cases}\sum_{i=1}^{k_1}\pi(S-r,i) \text{ for } s+1 \le r \le S.\\ \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,S-r,i,j), \text{for } 1 \le r \le s\end{cases}$

P (Inventory level=0, demand length L =0) = $\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,S,i,j)$

(3) The expected demand length E (L) can be calculated as follows. Demand length L = 0 when there is stock in the inventory or when the inventory becomes empty without a waiting demand. So for E(L) it can be seen that E(L)=$\sum_{k=S+1}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,\mathrm{k,j,i})(\mathrm{k-S})+\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(n,k,i,j)$ (nM+k-S).

Let δ_1= (0,0,…0,1,1,…1,2,2,…2,…,M-1-S,M-1-S,…,M-1-S)'be a column of type [(S-s) k_1+(M-S+s) k_1k_2] x 1 and in the vector, the number 0 appears [(S-s) k_1+(s+1) k_1k_2] times, and the numbers 1,2,3,.., (M-1-S) appear k_1k_2 times one by one in order. The vector δ_2= (0,0,…0,1,1,…1,2,2,…2,…,M-1, M-1, …, M-1)' where the all numbers 0 to M-1 appear k_1k_2 times. On simplification using equations (25)-(28)

E (L) = $\pi_0\delta_1$ +M π_1(I-R)$^{-2}$e + π_1(I-R)$^{-1}\delta_2$- Sπ_1(I-R)$^{-1}$e (30)

(4) Variance of the demand length can be seen using VAR (L) = E (L^2) – E(L)^2. Let δ_3 be column vector $\delta_3 = [0,..0,1^2,\ldots 1^2\ 2^2,..2^2,\ldots(M-1-S)^2,\ldots(M-1-S)^2]'$ of type [(S-s) k_1+(M-S+s) k_1k_2] x 1 where the number 0 appears [(S-s) k_1+(s+1) k_1k_2]times, and the square of numbers 1,2,3,.., (M-1-S) appear k_1k_2 times one by one in order and let $\delta_4 = [0,..0,1^2,\ldots 1^2\ 2^2,..2^2,\ldots(M-1)^2,\ldots(M-1)^2]'$ of type Mk_1k_2x1 where the number 0 appears k_1k_2 times, and the square of the numbers 1,2,3,.., (M-1) appear k_1k_2 times one by one in order. It can be seen that the second moment, E(L^2)=$\sum_{k=S+1}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,\mathrm{k,i,j})(\mathrm{k-S})^2$ + $\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(n,k,i,j)[Mn+k-S]^2$. Using Binomial expansion in the second series it may be noted E(L^2)=$\pi_0\delta_3 + M^2[\sum_{n=1}^{\infty}n(n-1)\pi_n e + \sum_{n=1}^{\infty}n\,\pi_n e] + \sum_{n=1}^{\infty}\pi_n\,\delta_4$ + 2M $\sum_{n=1}^{\infty}n\,\pi_n\delta_2$ -2S $\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(n,k,i,j)\ (Mn+k)$ +$S^2\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(n,k,i,j)$. After simplification,

E(L^2)= $\pi_0\delta_3$+$M^2[\pi_1(I-R)^{-3}2R\,e+\pi_1(I-R)^{-2}e]$+$\pi_1(I-R)^{-1}\delta_4$ + 2M $\pi_1(I-R)^{-2}\delta_2$

-2S [M π_1(I-R)$^{-2}$e + π_1(I-R)$^{-1}\delta_2$] +$S^2\pi_1(I-R)^{-1}e$ (31)

Using (30) and (31) variance of L can be written.

(5) The above partition method may also be used to study the case of (S,s) inventory system in which the supply for an order is a finite valued discrete random variable by suitably redefining the matrices U_j for 1 ≤ j ≤ N as presented in Rama Ganesan, Ramshankar and Ramanarayanan for M/M/1 bulk queues [9] and for PH/PH/1 bulk queues [14].

III. MODEL (B) MAXIMUM DEMAND SIZE M < MAXIMUM SUPPLY SIZE N

The dual case of Model (A), namely the case, M < N is treated here. (When M =N both models are applicable and one can use any one of them.) The assumption (iv) of Model (A) is changed and all its other assumptions are retained.

3.1.Assumption

(iv). The maximum demand size M=$\max_{1\le i\le k_1} M_i$ is less than the maximum supply size N=$\max_{1\le j\le k_2} N_j$.

3.2.Analysis

Since this model is dual, the analysis is similar to that of Model (A). The differences are noted below. The state space of the chain is as follows presented in a similar way. The state of the system of the Markov chain X(t) is X(t) = {(k, i) : for 0 ≤ k ≤ S-s-1 and 1 ≤ i ≤ k_1)} U {(0, k, i, j) ; for S-s ≤ k ≤ N-1; 1 ≤ i ≤ k_1; 1 ≤ j ≤ k_2} U{(n, k, i, j): for 0 ≤ k ≤ N-1; 1 ≤ i ≤ k_1; 1 ≤ j ≤ k_2 and n ≥ 1}. (32)

The chain is in the state (k, i) when the number of stocks in the inventory is S –k for 0 ≤ k ≤ S-s-1 and the arrival phase is i for 1 ≤ i ≤ k_1. The chain is in the state (0, k, i, j) when the number of stocks in the inventory is S-k for S-s ≤ k ≤ S without any waiting demand, the arrival phase is i for 1 ≤ i ≤ k_1 and the order-supply phase is j for 1 ≤ j ≤ k_2. The chain is in the state (0, k, i, j) when the number of waiting demands is k-S for S+1 ≤ k ≤ N-1, arrival phase is i for 1 ≤ i ≤ k_1 and the order-supply phase is j for 1 ≤ j ≤ k_2.The chain is in the state (n, k, i, j) when the number of demands in the queue is n N + k- S, for 0 ≤ k ≤ N-1 and 1 ≤ n < ∞, arrival phase is i for 1 ≤ i ≤ k_1 and the order-supply phase is j for 1 ≤ j ≤ k_2. When the number of demands waiting in the system is r ≥ 1, then r is identified with the first two co-ordinates (n, k) where n is the quotient and k is the remainder for the division of r + S by N; r = N n + k - S for r ≥ 1, 0 ≤ n < ∞ and 0 ≤ k ≤ N-1.The infinitesimal generator Q_B of the model has the same block partitioned structure given for Model(A) but the inner matrices are of different types and orders with distinct inner settings.

$$Q_B=\begin{bmatrix} B'_1 & B'_0 & 0 & 0 & . & . & . & \cdots \\ B'_2 & A'_1 & A'_0 & 0 & . & . & . & \cdots \\ 0 & A'_2 & A'_1 & A'_0 & 0 & . & . & \cdots \\ 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{33}$$

In (33) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}$, ... For the partition purpose the states in the first two sets of (1) are combined. The vector $\underline{0}$ is of type 1 x [$k_1(S-s)+k_1k_2$(N-S+s)] and $\underline{n}$ is of type 1 x(k_1k_2N).

$\underline{0}$=((0,1),(0,2)…(0,k_1)…(S-s-1,1)…(S-s-1, k_1),(0,S-s,1,1),(0,S-s,1,2)…(0,S-s, k_1, k_2)…(0,S,1,1)…(0,S, k_1, k_2), (0, S+1,1,1) … (0,S+1,k_1, k_2)…(0,N-1,1,1)…(0,N-1,$k_1 k_2$)) and for n ≥ 1, $\underline{n}$ = ((n,0,1,1), (n,0,1, 2) … (n, 0,1,k_2),(n,0,2,1),(n,0,2,2)…(n,0,2,k_2),(n,0,3,1)…(n,0,k_1, k_2),(n,1,1,1)...(n,1,k_1, k_2),(n,2,1,1)…(n,2, k_1, k_2)…. (n,N-1,1,1),(n,N-1,1,2) …(n,N-1,1,k_2),(n,N-1,2,1)…(n,N-1,k_1, k_2)). The matrices $B'_1 and\, A'_1$ have negative diagonal elements. They are of orders $k_1(S-s)+k_1k_2$(N-S+s) and k_1k_2N respectively and their off diagonal elements are non- negative. The matrices $A'_0\ and A'_2$ have nonnegative elements and are of order k_1k_2N. The matrices $B'_0\ and\ B'_2$ have non-negative elements and are of types [$k_1(S-s)+k_1k_2$(N-S+s)] x (k_1k_2N) and (k_1k_2N) x [$k_1(S-s)+k_1k_2$(N-S+s)] respectively and they are given below. Using Model (A) for definitions of Λ_j Λ'_j $and \Lambda''_j, for\ 1\ \le j\ \le M\ , and\ U_j,\ U'_j, V_j\ \ for\ 1\ \le j\ \le N$ and letting Q'_1=T⊕S, the partitioning matrices are defined as follows.

$$A'_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 \end{bmatrix} \tag{34}$$

$$A'_2 = \begin{bmatrix} U_N & U_{N-1} & U_{N-2} & \cdots & U_3 & U_2 & U_1 \\ 0 & U_N & U_{N-1} & \cdots & U_4 & U_3 & U_2 \\ 0 & 0 & U_N & \cdots & U_5 & U_4 & U_3 \\ 0 & 0 & 0 & \ddots & U_6 & U_5 & U_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & U_N \end{bmatrix} \tag{35}$$

$$A'_1 = \begin{bmatrix} Q'_1 & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0 \\ U_1 & Q'_1 & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ U_2 & U_1 & Q'_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & Q'_1 & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M \\ U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q'_1 & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U_{N-M+1} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & Q'_1 & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{N-2} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-2} & \cdots & Q'_1 & \Lambda_1 \\ U_{N-1} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-1} & \cdots & U_1 & Q'_1 \end{bmatrix} \tag{36}$$

$B'_1 =$

$$\begin{bmatrix} T & \Lambda''_1 & \cdots & \Lambda''_{S-s-1} & \Lambda'_{S-s} & \Lambda'_{S-s+1} & \cdots & \Lambda'_M & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & T & \cdots & \Lambda''_{S-s-2} & \Lambda'_{S-s-1} & \Lambda'_{S-s} & \cdots & \Lambda'_{M-1} & \Lambda'_M & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \Lambda''_{S-s-3} & \Lambda'_{S-s-2} & \Lambda_{S-s-1} & \cdots & \Lambda'_{M-2} & \Lambda'_{M-1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_{M-(S-s)} & \Lambda'_{M-(S-s-1)} & \cdots & \Lambda'_M & 0 & \cdots & 0 & 0 \\ V_{S-s-1} & U'_{S-s-1} & \cdots & U'_1 & Q'_1 & \Lambda_1 & \cdots & \Lambda_{M-(S-s+1)} & \Lambda_{M-(S-s)} & \cdots & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ V_{S-s} & U'_{S-s} & \cdots & U'_2 & U_1 & Q'_1 & \cdots & \Lambda_{M-(S-s+2)} & \Lambda_{M-(S-s+1)} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{N-M-2} & U'_{N-M-2} & \cdots & U'_{N-M-S+s} & U_{N-M-S+s-1} & U_{N-M-S+s-2} & \cdots & Q'_1 & \Lambda_1 & \cdots & \Lambda_{S-s} & \Lambda_{S-s+1} & \cdots & \Lambda_{M-1} & \Lambda_M \\ V_{N-M-1} & U'_{N-M-1} & \cdots & U'_{N-M-S+s+1} & U_{N-M-S+s} & U_{N-M-S+s-1} & \cdots & U_1 & Q'_1 & \cdots & \Lambda_{S-s-1} & \Lambda_{S-s} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ V_{N-3} & U'_{N-3} & \cdots & U'_{N-S+s-1} & U_{N-S+s-2} & U_{N-S+s-3} & \cdots & U_{N-M-2} & U_{N-M-3} & \cdots & U_{N-M-S+s-2} & U_{N-M-S+s-3} & \cdots & Q'_1 & \Lambda_1 \\ V_{N-2} & U'_{N-2} & \cdots & U'_{N-S+s} & U_{N-S+s-1} & U_{N-S+s-2} & \cdots & U_{N-M-1} & U_{N-M-2} & \cdots & U_{N-M-S+s-1} & U_{N-M-S+s-2} & \cdots & U_1 & Q'_1 \end{bmatrix} \quad (37)$$

The matrices $B'_0$and $B'_2$are similar to A'_0 and A'_2. The zero matrices appearing in the first S-s row blocks of B'_0 are of type k_1 x ($k_1 k_2$) and after the S-s th row block the matrices blocks are of order $k_1 k_2$. The model presented in (37) and (38) considers the case when M < N-S+s and the blocks Λ'_1 falls inside B'_1.When M ≥ N-S+s, then from the row block i+1, Λ'_j falls outside B'_1 and Λ'_j is to be placed in B'_0up to the row block S-s in place of Λ_j where i=S-s-N+M , without changing other terms in (34) .The matrix B'_2 is similar to A'_2.In the first S-s column blocks U'_j appears instead of U_j.

$$B'_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 \end{bmatrix} \quad (38)$$

$$B'_2 = \begin{bmatrix} U'_N & \cdots & U'_{N-S+s+1} & U_{N-S+s} & \cdots & U_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & U'_N & U_{N-1} & \cdots & U_{S-s} \\ 0 & \cdots & 0 & U_N & \cdots & U_{S-s+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & U_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & U_N \end{bmatrix} \quad (39)$$

The basic generator which is concerned with only the demand and supply is $Q''_B = A'_0 + A'_1 + A'_2$ and is presented in (40). This is also block circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is $\left(\frac{\varphi\otimes\phi}{N}, \frac{\varphi\otimes\phi}{N}, \frac{\varphi\otimes\phi}{N}, \ldots\ldots, \frac{\varphi\otimes\phi}{N}\right)$ and the stability condition remains the same. Following the arguments given for Model (A), one can find the stationary probability vector for Model (B) also in matrix geometric form. All performance measures including expectation of demands waiting for supply and its variance for Model (B) have the form as in Model (A) except M is replaced by N

$$Q''_B = \begin{bmatrix} Q'_1 + U_N & \Lambda_1 + U_{N-1} & \cdots & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 \\ U_1 & Q'_1 + U_N & \cdots & \Lambda_{M-2} + U_{N-M+2} & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_{N-M-2} & U_{N-M-3} & \cdots & Q'_1 + U_N & \Lambda_1 + U_{N-1} & \Lambda_2 + U_{N-2} & \cdots & \Lambda_M + U_{N-M} & U_{N-M-1} \\ U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q'_1 + U_N & \Lambda_1 + U_{N-1} & \cdots & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} \\ \Lambda_M + U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & Q'_1 + U_N & \cdots & \Lambda_{M-2} + U_{N-M+2} & \Lambda_{M-1} + U_{N-M+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 + U_{N-2} & \Lambda_3 + U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & Q'_1 + U_N & \Lambda_1 + U_{N-1} \\ \Lambda_1 + U_{N-1} & \Lambda_2 + U_{N-2} & \cdots & \Lambda_M + U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q'_1 + U_N \end{bmatrix} \quad (40)$$

IV. NUMERICAL CASES

For numerical illustration it is considered that the demand arrival time PH distribution has representation T= $\begin{bmatrix} -3 & 1 & 1 \\ 1 & -4 & 1 \\ 2 & 1 & -5 \end{bmatrix}$ with $\underline{\alpha}$ = (.3, .3, .4) and the service time PH distribution has representation S= $\begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$ with $\underline{\beta}$ = (.4, .6).

Six examples are studied. The maximum demand size and maximum supply size are fixed as M=8 and N=8 in examples 1 and 2. Examples 3 and 4 treat the cases M=8, N=7 and examples 5 and 6 treat the cases M=5, N=8.

The order of the rate matrix R is 48 since it is the product k_1k_2M and k_1k_2N depending on M >N or N > M. The probabilities of bulk demand sizes and bulk supply sizes are presented as follows in the examples.
In the examples 1 and 3 the bulk demand probabilities of sizes 1, 5 and 8 are ($p_{1,1}$=.5, $p_{1,5}$=.3, $p_{1,8}$=.2), ($p_{2,1}$=.4, $p_{2,5}$ =.4, $p_{2,8}$=.2), ($p_{3,1}$=.6, $p_{3,5}$=.3, $p_{3,8}$ =.1) in demand phases1, 2 and 3 respectively.
In the examples, 2 and 4 the bulk demand probabilities of sizes 1, 5 and 8 are ($p_{1,1}$=.5, $p_{1,5}$=.4, $p_{1,8}$=.1), ($p_{2,1}$=.4, $p_{2,5}$ =.5, $p_{2,8}$=.1), ($p_{3,1}$=.6, $p_{3,5}$=.4, $p_{3,8}$ =0) in demand phases1, 2 and 3 respectively.
In example 5, the bulk demand probabilities of sizes 1 and 5 are ($p_{1,1}$=.5, $p_{1,5}$=.5, $p_{1,8} = 0$), ($p_{2,1}$=.4, $p_{2,5}$=.6, $p_{2,8} = 0$), ($p_{3,1}$=.6, $p_{3,5}$ =4, $p_{2,8} = 0$) in demand phases1, 2 and 3 respectively.
In the example 6, the bulk demand probabilities of sizes 1, 5 and 8 are ($p_{1,1}$=.6, $p_{1,5}$=.4, $p_{1,8}$=0), ($p_{2,1}$=.5, $p_{2,5}$=.5, $p_{2,8}$=0), ($p_{3,1}$=.7, $p_{3,5}$=.3, $p_{3,8}$= 0) in demand phases1, 2 and 3 respectively. In all the examples, $p_{1,j} = p_{2,j} = p_{3,j}$ =0 for j = 2, 3, 4, 6 and 7.
The bulk supply sizes in examples 1, 2, 5 and 6 are $N_i = 8$, for PH phases i=1and 2 respectively. The bulk supply sizes in examples 3 and 4 are $N_i = 7$, for PH phases i=1and 2 respectively.
The iteration for the rate matrix R is performed for the same 15 number of times in all the six examples and the performance measures are written using the matrix iterated R(15) matrix of order 48. The difference norms of convergence are presented in table 1 along with expected demand lengths and the variances obtained for the examples. The inventory stock level probabilities, probability of both stock level and demand queue length are 0 and various block level probabilities are presented for the examples 1 to 6 in the table 1.
The performance measures obtained show significant variations depending on M, N and $p_{i,j}$. The two levels namely the inventory levels and block demand queue levels are both important in the inventory models. Their probability values are presented in figures 1 and 2.

Table1.Results obtained for six examples

Max Sizes	M=N=8	M=N=8	M=8, N=7	M=8, N=7	M=5, N=8	M=5, N=8
P(5)	0.10601487	0.11785438	0.08809618	0.10325321	0.12530646	0.12524953
P(4)	0.06563195	0.07308979	0.05437304	0.06389154	0.07779828	0.07786831
P(3)	0.05425577	0.06054094	0.04479202	0.05278723	0.06452293	0.06468091
P(2)	0.06177616	0.06321462	0.05795214	0.0619457	0.06394033	0.07541203
P(1)	0.04093185	0.04211368	0.0380935	0.04095292	0.04274263	0.05050615
P(S=0,L=0)	0.03690724	0.03821956	0.03402691	0.03683153	0.0389471	0.04611862
π0e	0.65385198	0.71748983	0.60488882	0.67795609	0.75952928	0.80962956
π1e	0.20360621	0.1801469	0.21408866	0.19341499	0.16093748	0.1401404
π2e	0.08249751	0.06595093	0.09663963	0.07800749	0.16093748	0.03802532
π3e	0.03470422	0.02338826	0.04498652	0.03062283	0.01648611	0.00913714
π4e	0.01464147	0.00836542	0.02099736	0.01209681	0.00523129	0.00230538
P(Block>4)	0.01069861	0.00465866	0.01839901	0.00790179	0.00235138	0.0007622
Norm	1.3822E-05	4.3169E-06	4.4917E-05	1.5885E-05	1.5738E-06	2.1682E-07
E(L)	4.12064594	3.07354874	5.18266342	3.75081133	2.48249547	1.77506724
VAR(L)	43.3269545	28.0508918	59.9366342	36.8113295	20.4977541	12.9188194
STD (L)	6.58232136	5.29630926	7.7418783	6.06723409	4.52744454	3.59427592

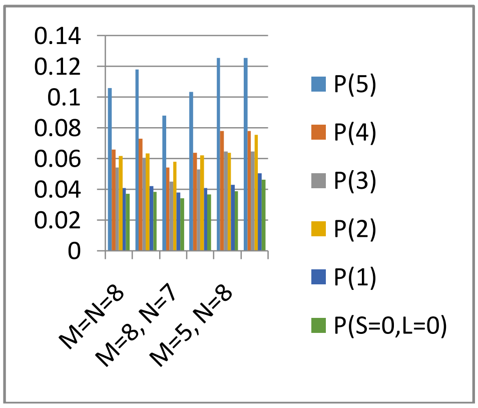

Figure 1 Probability of Inventory Levels

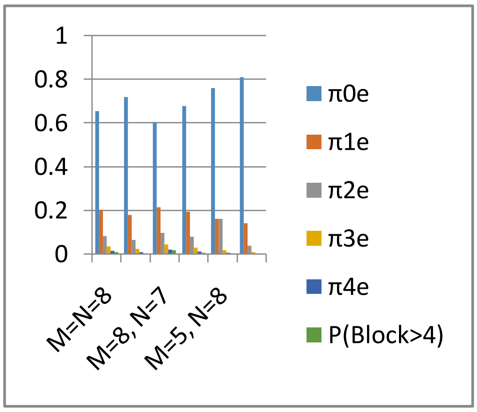

Figure2 Probability of Block Levels

V. CONCLUSION

Two (S, s) inventory systems with bulk demand and bulk supply after the lead time have been treated. Identifying the maximum of the demand size and supply size for forming the demand blocks of such maximum sizes along with PH phases, the stationary probability vectors are presented when the inter arrival times bulk demands and lead time distributions for supply have PH distributions. Matrix geometric solutions have been obtained by partitioning the infinitesimal generator by grouping of demands and PH phases together. The basic system generators of the bulk systems are block circulant matrices which are explicitly presenting the stability condition in standard forms. Numerical results for (S,s) inventory systems by varying bulk sizes are presented and discussed. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited. The PH distribution includes Exponential, Erlang, Hyper Exponential, and Coxian distributions as special cases and the PH distribution is also a best bet and approximation for a general distribution. Further the inventory systems with PH distributions are most general in forms and almost equivalent to inventory systems with general distributions. The bulk demand models because they have non-zero elements or blocks above the super diagonals in infinitesimal generators, they require for studies the decomposition methods with which queue length probabilities of the system are written in a recursive manner. Their applications are much limited compared to matrix geometric results. From the results obtained here, provided the maximum demand and supply sizes are not infinity, it is established that the most general model of the PH inventory system with bulk demand and bulk supply admits matrix geometric solution. Further studies with block circulant basic generator system may produce interesting and useful results in inventory theory and finite storage models like dam theory. It is also noticed here that once the maximum demand or supply size increases, the order of the rate matrix increases proportionally. However the matrix geometric structure is retained and rates of convergence is not much affected. Randomly varying environments causing changes in the sizes of the PH phases may produce further results if studied with suitable partition techniques.

VI. ACKNOWLEDGEMENT

The first author thanks ANSYS Inc., USA, for providing facilities. The contents of the article published are the responsibilities of the authors.

REFERENCES

[1] Salah, Rachid, Abdelhakim and Hamid, On the Approximation of Arbitrary Distributions by Phase-Type Distributions, www.ise.ncsu.edu/faculty/docs/IETrans.pdf

[2] Thangaraj.V and Ramanarayanan.R, (1983), An operating policy in inventory system with random lead times and unit demands, Math.Oper and Stat.Series Optimization, Vol.1 p111-124.

[3] Jacob.K.D and Ramanarayanan.R, (1988) An (S,s) Inventory System with Rest Periods to the Server, Naval Research Logistic Q, Vol 35 (1) pp119-123.

[4] D. Bini, G. Latouche, and B. Meini. (2005). Numerical methods for structured Markov chains, Oxford Univ. Press, Oxford.

[5] Chakravarthy.S.R and Neuts. M.F.(2014). Analysis of a multi-server queueing model with MAP arrivals of customers, SMPT, Vol 43, 79-95,

[6] Gaver, D., Jacobs, P and Latouche, G, (1984). Finite birth-and-death models in randomly changing environments. AAP.16, 715–731

[7] Latouche.G, and Ramaswami.V, (1998). Introduction to Matrix Analytic Methods in Stochastic Modeling, SIAM. Philadelphia.

[8] Neuts.M.F.(1981).Matrix-Geometric Solutions in Stochastic Models: An algorithmic Approach, The Johns Hopkins Press, Baltimore

[9] Rama Ganesan, Ramshankar and Ramanarayanan,(2014) M/M/1 Bulk Arrival and Bulk Service Queue with Randomly Varying Environment, IOSR-JM, Vol. 10, Issue 6,Ver.III, pp. 58-66.

[10] Sundar Viswanathan, (2014) Stochastic Analysis of Static and Fatigue Failures with Fluctuating Manpower and Business, IJCA, Vol. 106, No.4, Nov. pp.19-23.

[11] William J. Stewart, The matrix geometric / analytic methods for structured Markov Chains, N.C State University www.sti.uniurb/events/sfmo7pe/slides/Stewart-2 pdf.

[12] Rama Ganesan and Ramanarayanan (2014), Stochastic Analysis of Project Issues and Fixing its Cause by Project Team and Funding System Using Matrix Analytic Method, IJCA, Vol.107, No.7, pp.22-28.

[13] Sundar Viswanathan, Rama Ganesan, Ramshankar. R and Ramanarayaanan. R, (2015) Bulk Demand (S, s) Inventory System with Varying Environment , IOSR-JM ,Vol.11,Issue,Ver 1,Jan-Feb 2015 pp 75-82.

[14] Ramshankar.R, Rama Ganesan.R and Ramanarayanan. R, (2015), PH/PH/1 Bulk Arrival and Bulk Service Queue, IJCA, Vol. 109 (3) pp.27-33.